Polytropic Star

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Abstract

This note first discusses the density profile of a polytropic star. Then, it is shown that a radiative outer envelope of a star can be roughly described by a polytropic equation of state with $\gamma \simeq 4/3$. Finally, the negligible-mass outer layers of the star is studied.

Lane-Emden Equation

Defining $\rho = \rho_c\theta^n$, $\gamma = 1 + 1/n$, $P = K\rho^\gamma = K\rho_c^{1+1/n}\theta^{n+1}$, $\alpha^2 = (n+1)K\rho_c^{1/n-1}/(4\pi G)$, and $\xi = r/\alpha$, the equation of hydrostatic equilibrium ($\mathrm{d}P/\mathrm{d}r = -4\pi G\rho m(r)/r^2$) and mass continuity ($\mathrm{d}m/\mathrm{d}r = 4\pi\rho r^2$) can be written in the following Lane-Emden form

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{1}$$

The boundary conditions are $\theta(\xi = 0) = 1$ and $\mathrm{d}\theta/\mathrm{d}\xi(\xi = 0) = 0$ (since $\mathrm{d}P/\mathrm{d}r = 0$ at $r = 0$).

The above equation can be solved by defining a vector $\vec{X} = (\theta, \phi)^\mathrm{T}$, where $\phi \equiv -\xi^2\mathrm{d}\theta/\mathrm{d}\xi$. The spatial gradient of $\vec{X}$ is given by

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}\xi} = (-\phi/\xi^2, \xi^2\theta^n)^\mathrm{T}, \tag{2}$$

and the boundary condition is $\vec{X}(\xi = 0) = (1, 0)$. We integrate the above vector equation from $\xi = 0$ to a maximum ξ_max where θ has decreased to 0. It can be seen that $\phi > 0$ is a monotonically increasing function of ξ and $\theta(\xi)$ is a monotonically decreasing function.

It should be noted that since $\mathrm{d}\phi/\mathrm{d}\xi = \xi^2\theta^n \propto r^2\rho \propto \mathrm{d}m/\mathrm{d}r$, $\phi(\xi)$ is proportional to the mass coordinate:

$$\phi(\xi) = \frac{m(r)}{4\pi\alpha^3\rho_c}. \tag{3}$$

Once the functions $\theta(\xi)$ and $\phi(\xi)$ have been numerically obtained (including the value of ξ_max), we can solve for the two constants ρ_c and K from the physical stellar mass M and radius R by

$$\phi_\mathrm{max} \equiv \phi(\xi_\mathrm{max}) = \frac{M}{4\pi\alpha^3\rho_c},\ \alpha\xi_\mathrm{max} = R. \tag{4}$$

The results are

$$\rho_c = \frac{M/\phi_\mathrm{max}}{4\pi(R/\xi_\mathrm{max})^3},\ K = \frac{G}{n+1}(4\pi)^{1/n}(M/\phi_\mathrm{max})^{1-1/n}(R/\xi_\mathrm{max})^{3/n-1}. \tag{5}$$

The moment of inertia of a spherical shell of thickness $\mathrm{d}r$ is $\mathrm{d}I = (8\pi/3)\rho(r)r^4\mathrm{d}r$, which can be obtained by differentiating $I = (2/5)MR^2$ for a uniform sphere. Then, the moment of inertia of the entire star is given by

$$I = \frac{8\pi}{3}\rho_c\alpha^5\tilde{I} = \frac{2}{3}\frac{MR^2}{\phi_\mathrm{max}\xi_\mathrm{max}^2}\tilde{I}, \tag{6}$$

where

$$\tilde{I} \equiv \int_0^{\xi_{\rm max}} \theta^n \xi^4 {\rm d}\xi. \tag{7}$$

It is common to define a convenient constant

$$k = (2/3)\tilde{I}/(\phi_{\rm max}\xi_{\rm max}^2), \tag{8}$$

such that $I = kMR^2$.

The central potential of the star is given by the sum of the contribution from all spherical shells[1]

$$\Phi_{\rm c} = -G\int_0^R \frac{{\rm d}m}{r} = -4\pi\alpha^2 G\rho_{\rm c}\int_0^{\xi_{\rm max}} \frac{{\rm d}\phi}{\xi} = -\frac{GM}{R}\frac{\xi_{\rm max}}{\phi_{\rm max}}\tilde{\Phi}_{\rm c}, \tag{9}$$

where we have used ${\rm d}\phi/{\rm d}\xi = \xi^2\theta^n$, $M/R = 4\pi\alpha^2\rho_{\rm c}\phi_{\rm max}/\xi_{\rm max}$, and

$$\tilde{\Phi}_{\rm c} = \int_0^{\xi_{\rm max}} \theta^n \xi {\rm d}\xi. \tag{10}$$

Polynomial fits to the numerical results are

$$\begin{aligned} \xi_{\rm max} &\approx 0.05583n^4 - 0.2165n^3 + 0.5656n^2 + 0.1914n + 2.544, \\ \phi_{\rm max} &\approx 0.02393n^4 - 0.2363n^3 + 0.989n^2 - 2.402n + 4.767, \\ k = I/(MR^2) &\approx 2.112\times 10^{-4}n^4 - 1.969\times 10^{-3}n^3 + 0.02006n^2 - 0.1561n + 0.3992, \\ \Phi_{\rm c}/(-GM/R) &\approx 0.03297n^4 - 0.1246n^3 + 0.3245n^2 + 0.2092n + 1.557, \end{aligned} \tag{11}$$

which are accurate to fractional errors $< 0.2\%$ for $n \in (0.5, 3)$ (the above k fit has maximum fractional error of 2×10^{-4}).

The gravitational potential energy of the star is given by

$$U = -G\int_0^M \frac{m(r){\rm d}m}{r} = -\frac{GM^2}{R}\frac{\xi_{\rm max}}{\phi_{\rm max}^2}\tilde{U}, \tag{12}$$

where

$$\tilde{U} = \int_0^{\xi_{\rm max}} \phi(\xi)\theta^n \xi {\rm d}\xi. \tag{13}$$

The specific thermal energy is given by $e = P/[\rho(\gamma - 1)]$, so the total thermal energy is

$$T = \int_0^M e\,{\rm d}m = \frac{n}{n+1}\frac{GM^2}{R}\frac{\xi_{\rm max}}{\phi_{\rm max}^2}\tilde{T}, \tag{14}$$

where

$$\tilde{T} = \int_0^{\xi_{\rm max}} \theta^{n+1}\xi^2 {\rm d}\xi. \tag{15}$$

[1]Remember that the potential inside a spherical shell of mass ${\rm d}m$ and radius r is constant and equal to the surface value of $-G{\rm d}m/r$.

Miraculously, the condition of hydrostatic equilibrium dictates the potential and thermal energies of a star to be (cf. Chandrasekhar)

$$U = -\frac{3}{5-n}\frac{GM^2}{R},\ \ T = \frac{n}{5-n}\frac{GM^2}{R} \ \ \Rightarrow \ \ \frac{3T}{n} + U = 0 \text{ (virial equilibrium)}. \tag{16}$$

The total binding energy of a star is (physical systems correspond to $n < 3$)

$$U + T = -\frac{3-n}{5-n}\frac{GM^2}{R}. \tag{17}$$

Finally, let us consider how the star's radius responds as it loses mass rapidly such that the entropy profile of the remaining layers stays unchanged. Fixing the polytropic index n and enforcing $\mathrm{d}K = 0$, we obtain

$$\frac{\mathrm{d}\ln R}{\mathrm{d}\ln M} = \frac{n-3}{n-1} = \frac{2-\gamma}{4-3\gamma}. \tag{18}$$

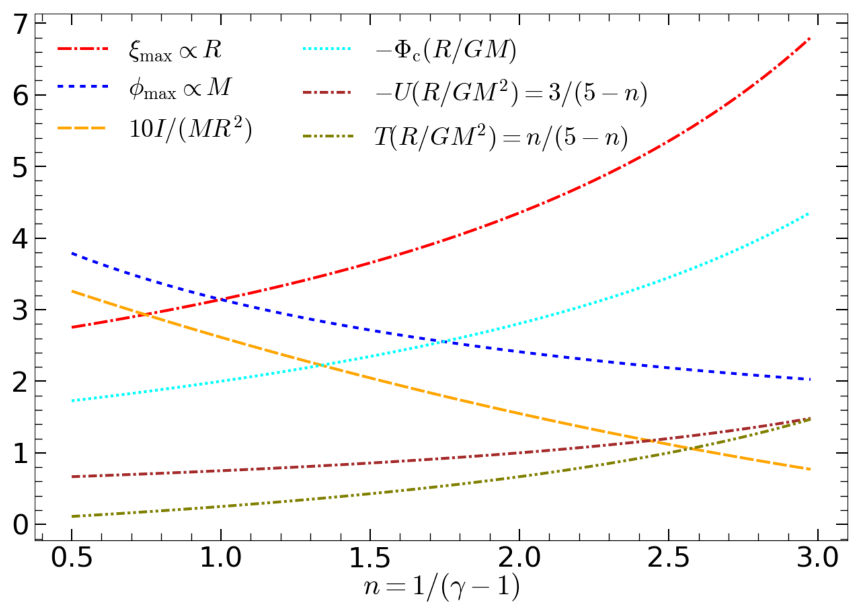

Fig. 1.— The integral quantities $\xi_{\max}$ (related to stellar radius), $\phi_{\max}$ (related to stellar mass), $k \equiv I/(MR^2)$ (related to the moment of inertia), Φ_c (central potential), U (gravitational potential energy), and T (thermal energy), for a polytropic star. Here, we show $10k$ (instead of k) for clarity.

Radiative Envelope

The polytropic equation of state works well for a convective envelope where the entropy is constant — and we have $P \propto \rho^{5/3}$ across the entire convective envelope.

However, for a radiative envelope, one needs to consider the radiative diffusion

$$\frac{L}{4\pi r^2} = -\frac{c}{3\rho\kappa}\frac{\mathrm{d}(aT^4)}{\mathrm{d}r}. \tag{19}$$

This should be combined with the equation for hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{GM}{r^2}. \tag{20}$$

Let us assume that the pressure is dominated by the gas (instead of radiation), which means $P \simeq \rho kT/(\mu m_{\mathrm{p}})$. This allows us to elliminate ρ in the above two equations. Taking the ratio between these two equations, we obtain

$$\frac{T^3}{\kappa}\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{3L}{16\pi acGM}. \tag{21}$$

Since we are considering the outer envelope, the right-hand side of the above equation is nearly a constant.

Let us then consider a particular form of Rosseland-mean opacity $\kappa = \kappa_0 P^q T^{-q-s}$, and for Kramer's law, one has $q = 1$ and $s = 3.5$. Then, the relation between T and P is given by

$$\mathrm{d}T^{4+q+s} = C\,\mathrm{d}P^{q+1},\ C = \frac{4+q+s}{q+1}\frac{3\kappa_0 L}{16\pi acGM}. \tag{22}$$

One can integrate the above equation from the outer boundary (near the photosphere) where $T = T_{\mathrm{ph}}$ and $P = P_{\mathrm{ph}}$. Then, in the regions far below the photosphere where $T \gg T_{\mathrm{ph}}$ and $P \gg P_{\mathrm{ph}}$, one must obtain the scaling

$$T^{4+q+s} \propto P^{q+1}, \tag{23}$$

which means $P \propto \rho^{1+(q+1)/(s+3)}$. For Kramer's law ($q = 1$ and $s = 3.5$), we have $P \propto \rho^{1.31}$, which is very close to the case of a $\gamma = 4/3$ polytrope. The proportional constant $K = P/\rho^\gamma$ depends on the stellar luminosity and mass. This applies to main-sequence stars with masses greater than about $1M_\odot$ (for lower-mass stars, the envelope becomes convective due to partial ionization of H).

In the extreme limit where the pressure is dominated by radiation $P \simeq aT^4/3$ (although this is usually not fully realized), then the radiative transfer equation can be written as

$$\rho\frac{L}{4\pi r^2} = -\frac{c}{\kappa}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{24}$$

Taking the ratio between the above equation and the hydrostatic equilibrium equation, one obtains

$$L = \frac{4\pi GMc}{\kappa} = L_{\mathrm{Edd}}. \tag{25}$$

In order to maintain locally Eddington luminosity everywhere, the opacity must be nearly constant over the envelope. This can be achieved if κ is dominated by electron scattering. In deeper layers where $\kappa = \kappa_0\rho^q T^{-s}$, one must have $T \propto \rho^{q/s}$ and hence $P \propto \rho^{4q/s}$. For Kramer's law ($q = 1$ and $s = 3.5$), we have $P \propto \rho^{1.14}$ — a radiation pressure-dominated envelope is expected to be highly centrally concentrated.

Structure of the outer layers

The structure of the outermost layers of the star is very simple when we ignore the self-gravity. From hydrostatic equilibrium and the equation of state, we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2}, P = K\rho^\gamma \implies \mathrm{d}\rho^{\gamma-1} = \frac{(\gamma-1)GM}{\gamma K}\mathrm{d}r^{-1}. \tag{26}$$

This can be easily integrated using the boundary condition of $\rho(r = R) = 0$,

$$\rho^{\gamma-1}(r) = \frac{(\gamma-1)GM}{\gamma K}\left(\frac{1}{r} - \frac{1}{R}\right), \text{ or } \rho(r) = \left[\frac{GM}{(n+1)K}\left(\frac{1}{r} - \frac{1}{R}\right)\right]^n. \tag{27}$$

The pressure profile is given by

$$P(r) = K\left[\frac{GM}{(n+1)K}\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{n+1}. \tag{28}$$

Using K in eq. (5), we obtain

$$P(r) = \frac{GM^2}{4\pi R^4}\frac{(R/r-1)^{n+1}}{n+1}\phi_{\max}^{n-1}\xi_{\max}^{3-n}. \tag{29}$$

This means that the pressure scaleheight near the surface is of the order $H \sim R - r$ (for $r \approx R$). Another consequence is that the isothermal sound speed $c_\mathrm{s} \equiv \sqrt{P/\rho}$ has the following simple form

$$c_\mathrm{s}^2 = K\rho^{\gamma-1} = \frac{GM\,(R-r)}{(n+1)rR}. \tag{30}$$

The exterior mass is given by

$$\begin{aligned} M_\mathrm{ex}(r) &= \int_r^R 4\pi r^2 \rho(r)\mathrm{d}r \\ &= 4\pi\left(\frac{(\gamma-1)GM}{\gamma K}\right)^{\frac{1}{\gamma-1}} R^{\frac{3\gamma-4}{\gamma-1}} \int_{r/R}^1 x^{\frac{2\gamma-3}{\gamma-1}}(1-x)^{\frac{1}{\gamma-1}}\mathrm{d}x, \end{aligned} \tag{31}$$

or

$$M_\mathrm{ex}(r) = 4\pi\left(\frac{GM}{(n+1)K}\right)^n R^{3-n}\left[B(3-n, n+1, 1) - B(3-n, n+1, r/R)\right], \tag{32}$$

where $B(a, b, x) = \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t$ is the incomplete Beta-function. Using the entropy constant K in eq. (5), we obtain

$$\frac{M_\mathrm{ex}(r)}{M} = \phi_{\max}^{n-1}\xi_{\max}^{3-n}\left[B(3-n, n+1, 1) - B(3-n, n+1, r/R)\right]. \tag{33}$$

In the limit $r/R \approx 1$, this can be further simplified into

$$\frac{M_\mathrm{ex}(r)}{M} \approx \phi_{\max}^{n-1}\xi_{\max}^{3-n}\frac{(1-r/R)^{n+1}}{n+1}, \text{ for } 1 - r/R \ll 1. \tag{34}$$

In the same limit, the pressure profile is given by

$$\frac{P(r)}{\bar{P}} \approx \phi_{\rm max}^{n-1}\xi_{\rm max}^{3-n}\frac{(1-r/R)^{n+1}}{n+1}, \text{ for } 1-r/R \ll 1, \tag{35}$$

where $\bar{P} \equiv GM^2/(4\pi R^4)$ is a rough estimate of the mean pressure inside the star. Thus, we have arrived at the following interesting result

$$\frac{P(r)}{\bar{P}} = \frac{M_{\rm ex}(P)}{M}, \tag{36}$$

where $M_{\rm ex}(P)$ is the mass exterior to a critical radius specified by a given pressure P.